Laboratory test of the MOND law

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1. Introduction

MOND expressed as a modified gravitational potential

The piecewise form of the MOND law is

$$\mathbf{F} = m\mathbf{a} = \begin{cases} -\dfrac{GmM}{r^2}\hat{\mathbf{r}} & \dfrac{GM}{r^2} \gg a_0 \\ -\dfrac{m\sqrt{a_0 GM}}{r}\hat{\mathbf{r}} & \dfrac{GM}{r^2} \ll a_0 \end{cases} . \tag{1}$$

Write this as

$$\mathbf{F} = m\mathbf{a} = -\nabla\phi \tag{2}$$

where ϕ is the gravitational potential acting on a test mass m due to a source mass M. Comparing these two gives

$$\phi = \begin{cases} -\dfrac{GmM}{r} & \dfrac{GM}{r^2} \gg a_0 \\ m\sqrt{a_0 GM}\log r & \dfrac{GM}{r^2} \ll a_0 \end{cases} . \tag{3}$$

Energy conservation

A sufficient condition that a force is conservative is that it can be written as a gradient of a scalar function, as it is here. In that case, irrespective of the function ϕ, one can always compute the non-relativistic energy by multiplying by the velocity and then integrating over time. One has

$$m\mathbf{a}.\mathbf{v} + \mathbf{v}.\nabla\phi = 0 \Rightarrow \frac{m}{2}\frac{d\mathbf{v}^2}{dt} = -\frac{d\phi}{dt} \tag{4}$$

which expresses the fact of total power conservation. Therefore

$$\frac{m\mathbf{v}^2}{2} + \phi = E_0 \tag{5}$$

where E_0 is a constant. That is, the KE + PE is a (conserved) constant. So energy is conserved: the final kinetic energy is due to loss of potential energy.

2. Superposition

Newtonian limit

A difficulty comes in trying to figure out how to think about each of the constituent parts of a mass, since the rule (3) is inconsistent with super-position of masses *M*. Usually one has

$$\phi_{total}(\mathbf{r}) = \sum_i \phi_i(\mathbf{r}) = -Gm\sum_i \frac{M_i}{|\mathbf{r}-\mathbf{r}_i|} \tag{6}$$

that is, the potentials of individual masses add. Consequently, the effective gravitational mass of a clump of masses is the sum of the masses

$$\phi_{total}(\mathbf{r}) = -\frac{GmM_{total}}{r}; \quad M_{total} = \sum_i M_i \tag{7}$$

provided one is sufficiently distant from the masses. In particular, if for example all the M_i are equal, then (7) follows from (6) only if

$$r \gg |\mathbf{r}_i - \mathbf{r}_j| \forall i \neq j \tag{8}$$

whereupon

$$\phi_{total}(\mathbf{r}) = -\frac{GmM_{total}}{r}; \quad M_{total} = NM \tag{9}$$

where *N* is the number of masses in the clump. Eq. (8) is just the requirement that the clump of masses subtend a small angle at the location of the test mass. If so, then the clump can be considered a single mass whose value is a linear superposition of its component masses.

General case

To help keep the notation simple, let us assume that MOND is properly described by the continuous force law

$$\mathbf{F} = m\mathbf{a} = -m\hat{\mathbf{r}}\left(\frac{\sqrt{a_0 GM}}{r} + \frac{GM}{r^2}\right) \tag{10}$$

which has the desired properties of (1) away from the knee. Let us normalize all distances with respect to the distance at which the force goes over from Newtonian to MOND, i.e. at $r_0 = \sqrt{GM/a_0}$ by using $\overline{r} = r\sqrt{a_0/GM}$. Then (10) is

$$\mathbf{F} = m\mathbf{a} = -ma_0\hat{\mathbf{r}}\left(\frac{1}{\overline{r}} + \frac{1}{\overline{r}^2}\right). \tag{11}$$

The corresponding potential is then

$$\phi = ma_0 r_0\left(\log\overline{r} - \frac{1}{\overline{r}}\right) = m\sqrt{GMa_0}\left(\log\overline{r} - \frac{1}{\overline{r}}\right). \tag{12}$$

Since forces and therefore potentials add, in place of (6) one now has that the effective potential felt by a test mass is

$$\phi_{total}(\mathbf{r}) = \sum_i \phi_i(\mathbf{r}) = m\sqrt{Ga_0}\sum_i\left\{\sqrt{M_i}\left(\log\overline{r}_i - \frac{1}{\overline{r}_i}\right)\right\}. \tag{13}$$

Again, for the sake of simplicity, let us assume that the distribution of masses is such that each source mass is either distinctly 'Newtonian' or 'MOND like' in it potential-producing effects:

$$\left|\log \bar{r}_i\right| \gg 1/\bar{r}_i, \quad \left(\bar{r}_i \gg 1\right) \text{ OR } \left|\log \bar{r}_i\right| \ll 1/\bar{r}_i, \quad \left(\bar{r}_i \ll 1\right). \tag{14}$$

Note that this is not quite the same as assuming the same condition on the force-producing effects, but the difference is unimportant if the inequality is well-satisfied. Then the sum (13) splits cleanly into two parts:

$$\phi_{total}\left(\mathbf{r}\right) = m\sqrt{Ga_0}\left(\sum_{i=1}^{N_2}\sqrt{M_i}\log \bar{r}_i - \sum_{i=1}^{N_1}\frac{\sqrt{M_i}}{\bar{r}_i}\right) = m\left(\sum_{i=1}^{N_2}\sqrt{Ga_0 M_i}\log\frac{r_i}{\sqrt{GM_i/a_0}} - \sum_{i=1}^{N_1}\frac{GM_i}{r_i}\right) \tag{15}$$

where in the second expression we have restored the units so that we can see the masses M_i explicitly. Now assume we can split the aggregate masses up into equal-sized pieces each of mass M:

$$\phi_{total}\left(\mathbf{r}\right) = m\left(\sum_{i=1}^{N_2}\sqrt{GMa_0}\log r_i - GM\sum_{i=1}^{N_1}\frac{1}{r_i}\right) + const = m\left(N_2\sqrt{GMa_0}\left\langle \log r_{MOND}\right\rangle - GMN_1\left\langle\frac{1}{r_{Newton}}\right\rangle\right) + const\,. \tag{16}$$

Following the argument above, if each group of masses is sufficiently clumped (the constraints on the MOND clump are different from those on the Newtonian clump) then the average of the function is the same as the function of the average, and

$$\phi_{total}\left(\mathbf{r}\right) = m\left(N_2\sqrt{GMa_0}\log\left\langle r_{MOND}\right\rangle - GMN_1\frac{1}{\left\langle r_{Newton}\right\rangle}\right) + const \tag{17}$$

where the MOND and Newton radii are the nominal distances of the two clumps. Granted these considerations, the Newtonian regime source masses can be superposed in the normal way, and the MOND regime source masses can be superposed separately is if the appropriate strength of each unit is $\sqrt{M}$ rather than M.

According to (17) and more generally (13), distant masses will contribute in the MOND regime, long after their Newtonian influence has become negligible.

3. Experimental determination of MOND when EM forces are also present

Earthtech Experiment

In the experiment at Earthtech it is not possible to eliminate the effect of EM forces. That is, when the Cavendish test mass is static, one does not know the relative contributions of gravitational and EM forces, and it is not possible to estimate them with any confidence. We seek therefore a plausible prediction for the behavior of a mass when in the MOND regime of a source mass whilst significant EM sources are present.

If the conjecture of superposition described above applies, then a static test mass must be regarded as immersed in a sum of three different types of forces: I) Newtonian gravitational forces, II) MOND regime forces, and III) EM forces (such as from the torsion from the suspending fiber). Together these will produce an effective *background potential* $\phi_{backgnd}\left(\mathbf{r}\right)$ which a priori is not known.

In the first - bootstrap - stage of the experiment, this profile must be measured empirically by applying a force whose profile is known exactly. In our case the latter should be a Newtonian gravitational force from a sufficiently large nearby source mass. If we are lucky this test force will induce simple harmonic motion as predicted by the source mass alone exactly in accord with the absence of a background. But if not, then the departure from expectation will reveal the background, whose cause must remain unknown. The background force can be regarded as constant only if its variation is over the range of motion of the Cavendish balance is much smaller than the MOND force we intend to apply in the second stage. If not, then the quadratic and possibly higher order variation of the background force must be computed in the bootstrap phase.

In the second stage of the experiment the source mass should be chosen so that the new potential *departs from background* by the MOND-predicted value and the acceleration curve measured accordingly.

Abramovici experiment

Abramovici and Vager (Phys. Rev. D, Vol. 34 No. 10, 3240-3241, 1986) designed an experiment to measure a MOND-type modification to Newton's second law. Their assumption was that astrophysical observations of MOND-like behavior are due not specifically to a change in Newton's law of gravitation, but to a change in Newton's second law (i.e. the left hand side of (1)). Such a modification appears to be problematic however, since the derivation of the conservation of energy (see above) relies on the fact that the left hand side of (1) can be converted to a total derivative with respect to time by multiplication by a velocity (i.e. to give a power) – whose integral therefore is a constant (i.e. the energy). There is no other form of acceleration dependence that has this property. In any case, consistent with the hypothesis they wished to test, Abramovici and Vager used a small electric field (rather than a gravitational-field) to accelerate a test mass around the MOND level of a_0. They found no departure from Newtonian behavior. Their results are consistent with the possibility that MOND is a modification of the law giving the force (versus distance) rather than the law giving the response (Newton's second law). In particular their result is consistent with a modification of Newton's law of gravity along the lines of (1). [Their result does however appear to rule out a MOND-like modification to the EM Lorentz force.]